Termination for Direct Sums of Left-Linear Complete Term Rewriting Systems

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Abstract. A term rewriting system is called complete if it is confluent and terminating. We prove that completeness of TRSs is a "modular" property (meaning that it stays preserved under direct sums), provided the constituent TRSs are left-linear. Here, the direct sum $R_0 \oplus R_1$ is the union of TRSs R_0, R_1 with disjoint signature. The proof hinges crucially upon the (non)deterministic collapsing behavior of terms from the sum TRS.

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1. *Introduction*

An important concern in building algebraic specifications is their hierarchical or modular structure. The same holds for term rewriting systems (see Huet & Oppen [1980], Klop [1992], or Dershowitz and Jouannaud [1990]), which can be viewed as implementations of equational algebraic specifications. Specifically, it is of obvious interest to determine which properties of term rewriting systems

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(TRSs) have a "modular" character. We call a property *modular* if its validity for a TRS, hierarchically composed of some smaller TRSs, can be inferred from the validity of that property for the constituent TRSs. Naturally, the first step in such an investigation considers the most basic properties of TRSs: confluence, termination, unique normal form property, and similar fundamental properties as well as combinations thereof.

As to the modular structure of TRSs, it is again natural to consider as a start the most simple way that TRSs can be combined to form a larger TRS: namely, as a disjoint sum. This means that the alphabets of the TRSs to be combined are disjoint, and that the rewrite rules of the sum TRS are the rules of the summand TRSs together. (Without the disjointness requirement, the situation is even more complicated—for some results in this direction, see Dershowitz [1981], Toyama [1988], and Middeldorp and Toyama [1991].) A disjoint union of two TRSs R_0, R_1 is called in our paper, a *direct sum*, notation $R_0 \oplus R_1$.

Another simplifying assumption that we will make is that R_0, R_1 are homogeneous TRSs, that is, their signature is one-sorted (as opposed to the many-sorted or heterogeneous case; for results about direct sums of heterogeneous TRSs, see Ganzinger and Giegerich [1987]).

The first result in this setting is due to Toyama [1987a], where it is proved that confluence is a modular property. That is, $R_0 \oplus R_1$ is confluent $\Leftrightarrow$ R_0 and R_1 are confluent. Here "$\Rightarrow$" is trivial; "$\Leftarrow$" is what we are interested in. (For a simplified proof, see Klop et al. [1994].) To appreciate the nontriviality of this fact, it may be contrasted with the fact that another fundamental property, termination, is *not* modular, as the following simple counterexample in Toyama [1987b] shows:

$$R_0 = \{F(0,1,x) \to F(x,x,x)\}$$
$$R_1 = \{G(x,y) \to x, G(x,y) \to y\}.$$

It is trivial that R_0 and R_1 are terminating. However, $R_0 \oplus R_1$ is not terminating, because $R_0 \oplus R_1$ has the infinite reduction sequence:

$$F(G(0,1),G(0,1),G(0,1)) \to F(0,G(0,1),G(0,1)) \to F(0,1,G(0,1)))$$
$$\to F(G(0,1),G(0,1),G(0,1)) \to \cdots.$$

However, this counterexample uses a non-confluent TRS R_1. A more complicated counterexample to the modularity of "termination," involving only confluent TRSs, was given by J. W. Klop and H. P. Barendregt (private communication) (for ground terms only). (For this counterexample as well as for some improved versions, holding for open terms as well, and even using TRSs that are "irreducible," see Toyama [1987b].) Rephrased, this means that the important property of "completeness" of TRSs (a TRS is complete if it is both confluent and terminating) is not modular, that is, there are complete TRSs R_0, R_1 such that $R_0 \oplus R_1$ is not complete (in fact, not terminating; confluence of $R_0 \oplus R_1$ is ensured by the theorem in Toyama [1987a]). This counterexample, however, uses non-left-linear TRSs.

The point of the present paper is that left-linearity is essential; if we restrict ourselves to left-linear TRSs, then completeness is modular. Thus, we prove: If R_0, R_1 are left-linear (meaning that the rewrite rules have no repeated

variables in their left-hand sides), then $R_0 \oplus R_1$ is complete iff R_0, R_1 are. As left-linearity is a property that can be easily checked, and many equational algebraic specifications can be given by left-linear TRSs, we feel that this result is worthwhile.

The proof, however, is rather intricate and not easily digested. A crucial element in the proof, and in general in the way that the summand TRSs interact, is how terms may "collapse" to a subterm. The problem is that this collapsing behavior may exhibit a "nondeterministic" feature, which is caused by ambiguities among the rewrite rules. We hope that the present paper is of value not only because it establishes a result that in itself is simple enough, but also because of the analysis necessary for the proof that gives a kind of structure theory for disjoint combinations of TRSs and that may be of relevance to other similar studies.

Regarding the question of modular properties in the present simple set-up, we mention the results by Rusinowitch [1987] and Middeldorp [1989a]. These papers, together, contain a complete analysis of the cases in which termination for $R_0 \oplus R_1$ may be concluded from termination of R_0, R_1, depending on the distribution among R_0, R_1 of so-called collapsing and duplicating rules. Below, we will compare these results with ours.

Another useful fact is established in Middeldorp [1989b], where it is proved that the "unique normal form property" is a modular property. A survey of modularity results is in Middeldorp [1990]. For more modularity results, also pertaining to a special form of termination called *simple termination*, we refer to Kurihara and Kaji [1988/1990] and Kurihara and Ohuchi [1989/1990].

From now on we assume that the two TRSs R_0 and R_1 are terminating Without the assumption about the disjointness of R_0 and R_1, Bachmair and Dershowitz [1986] have shown: If R_0 is left-linear, R_1 is right-linear, and there is no overlap between left-hand sides of R_0 and right-hand sides of R_1, then the combined system $R_0 \cup R_1$ is terminating.

If R_0 and R_1 are disjoint and R_1 is *noncollapsing*, then nonoverlapping between left-hand sides of R_0 and right-hand sides of R_1 is trivial. Here, note that the noncollapsing property of R_1 is necessary since a variable right-hand side overlaps any left-hand side. Hence, from Bachmair and Dershowitz's theorem, it follows that if R_0 is left-linear, R_1 is right-linear and noncollapsing, then the direct sum $R_0 \oplus R_1$ is terminating. However, using the disjointness of R_0 and R_1 we can strengthen this result somewhat. Indeed, it holds that if R_1 is right-linear and noncollapsing, then the direct sum $R_0 \oplus R_1$ is terminating. This is a special case of Middeldorp's [1989a] theorem: If R_1 is noncollapsing and nonduplicating, then the direct sum $R_0 \oplus R_1$ is terminating. Here, the properties "noncollapsing" and "nonduplicating" (with right-linearity as a special case) of TRSs are seen to be basic criteria for termination.

The point of our result is that it shows that "left-linearity and confluence" is also a basic criterion for termination of the direct sum $R_0 \oplus R_1$. Indeed, in the present paper, we prove that, if R_0 and R_1 are left-linear and confluent, then the direct sum $R_0 \oplus R_1$ is terminating. Rusinowitch [1987] has proved:

(A) If R_0 and R_1 are noncollapsing, then the direct sum $R_0 \oplus R_1$ is terminating.
(B) If R_0 and R_1 are nonduplicating, then the direct sum $R_0 \oplus R_1$ is terminating.

So, a natural question from the observations above is whether "noncollapsing" (or "nonduplicating") and "left-linear and confluent" are interchangeable as follows:

(A′) If R_0 is noncollapsing and R_1 is left-linear and confluent, then the direct sum $R_0 \oplus R_1$ is terminating.
(B′) If R_0 is nonduplicating and R_1 is left-linear and confluent, then the direct sum $R_0 \oplus R_1$ is terminating.

We can prove that Conjecture A′ holds; the proof is in Appendix B. However, Conjecture B′ does not hold because of Toyama's [1987a] counterexample.

Finally, we mention why, in this paper, we cannot use the proof method developed by Breazu-Tannen and Gallier [1989] and Okada [1989] in their results about termination of the union of the (polymorphic) typed lambda calculus (λ) and a complete term rewriting system R. The reason is that their proof method is essentially analogous to the one for Conjecture A′ (see Appendix B)—but it does not work for our main result. Though, in their framework, R is not required to be noncollapsing, R actually plays the role of a noncollapsing system with respect to λ. The key fact in their proof is that R has base type but not higher type. Thus, if a mixed term M has a TRS context as outermost layer, then any term reduced from M has a base type. In particular, it will have no λ symbol at the root after any reduction. Hence, R-reductions cannot create a new β-redex in a mixed term, just like noncollapsing systems. It is clear that λ is left-linear and confluent. Thus, by taking R and λ as R_0 and R_1, respectively, in Conjecture A′, we can observe very similar proof diagrams when comparing their proof and our proof of Conjecture A′.

We now give a brief outline of the contents of the paper. Section 2 fixes some standard terminology and notation. Section 3 introduces an underlining device, necessary to keep track of subterms in a reduction. Section 4 considers "mixed terms," that is, terms obtained from the disjoint union of two TRSs. Several basic notions for mixed terms are introduced. The main lemma of this section (4.8.5) has a complicated proof, that is postponed to Appendix A. In Section 5, we prove the main result of this paper. In order not to interrupt the flow of arguments, the long proof of one crucial proposition (5.2) is deferred to Section 6.

1.1. PRELIMINARIES. We assume that the reader is familiar with the basic concepts and notations concerning term rewriting systems (TRSs); otherwise, see the basic references mentioned in the Introduction. In this section, we exhibit the notions and concepts that are specific for the present paper, and we briefly recapitulate some of the more basic concepts.

A term rewriting system R has an *alphabet* consisting of a (possibly infinite) set $\mathcal{F}$ of function symbols $F, G, H, \ldots$, each having an "arity," that is, the number of arguments that the function symbol requires, and a set of variables $x, y, z, \ldots$. So if F is n-ary, then $F(t_1, \ldots, t_n)$ is a term, for terms $t_1, \ldots, t_n$. Constants are 0-ary function symbols. The set of terms of R, notation $\mathrm{Ter}(R)$, contains the terms which are inductively generated from the constant symbols, the variables $x, y, z, \ldots$ and the other function symbols. Terms are denoted by $t, s, \ldots$ but occasionally also by $M, N, \ldots$.

A TRS R has a (possibly infinite) set Red(R) of *reduction* or *rewrite rules* r: $t \to s$, or $t \to_r s$, where r is the name of the rewrite rule. We adopt the usual restriction that the left-hand side t is not a variable, and that the variables occurring in s, also occur in t. A rewrite step has the form $C[t^\sigma] \to_r C[s^\sigma]$, where σ is a substitution and $C[\ ]$ a context, that is, a term with a "hole" $\Box$. A substitution σ is a homomorphism from Ter(R) to Ter(R), satisfying $F(t_1, \ldots, t_n)^\sigma = F(t_1^\sigma, \ldots, t_n^\sigma)$. The transitive reflexive closure of $\to_r$ is $\twoheadrightarrow_r$; the transitive closure of $\to_r$ is $\to_r^+$. The reflexive closure of $\to_r$ is $\to_r^=$. The convertibility (i.e., equivalence relation) generated by $\to_r$ is $=_r$. Often the subscript r is omitted. Convertibility ($=$) should not be confused with $\equiv$, which denotes syntactical equality. The notation $t \twoheadrightarrow^n s$ is short for $t \to \cdots \to s$ in n steps.

We say that t is a normal form if there is no s such that $t \to s$. Further, s has a normal form t if $s \twoheadrightarrow t$ and t is a normal form. The normal form of t is denoted by $t\!\downarrow$ if it exists uniquely.

The concepts of confluence (also called the Church–Rosser property) and termination (also called the Strong Normalization property, stating that all reduction sequences must end eventually) are defined as usual. We say that t is terminating if every reduction sequence starting with t eventually must terminate. A TRS is "*complete*" if its reduction relation is confluent and terminating (this is also called *canonical* in the literature). A TRS R is *left-linear* if R contains no rewrite rule $t \to s$ such that t contains two or more occurrences of the same variable.

We write $t \subseteq s$ to indicate that t is a subterm of s. When referring to subterms, we will always mean specific *occurrences* of those subterms; we will however not need a more precise formalism to indicate occurrences (e.g., as sequence numbers). If $t \subseteq s$ and $t \not\equiv s$, we write $t \subset s$, and call t a *proper* subterm of s.

2. *Underlined Reduction and Frozen Subterms*

Consider the TRS with set of reduction rules $\{F(x, C) \to x, F(C, x) \to x, H(x) \to x, G(x) \to x\}$ and the term $M \equiv F(H(C), G(C))$. Figure 1(a) displays the node-labeled tree corresponding to M. The term M has the following reductions to its normal form:

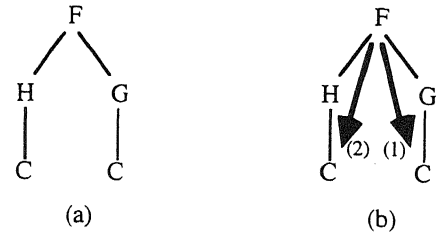
(1) $M \to F(C, G(C)) \to G(C) \to C$
(2) $M \to F(H(C), C) \to H(C) \to C$.

Although both reductions end in C, the two C's are different with respect to their occurrence in M. This is graphically expressed in Figure 1(b) where the arrows indicate to which occurrence of C the term M is "collapsed."
In the sequel, we will need to be precise about such reductions to *occurrences* of subterms, rather than mere subterms. Therefore, we introduce the concepts of "underlined" reductions and "frozen" subterms, as follows:

Definition 2.1

(i) Let R be some TRS. Then R_e is the TRS having as alphabet (or signature): the alphabet of R extended with a new unary function symbol 'e.' The rewrite rules of R_e are: those of R extended with $e(x) \to x$.

FIGURE 1



(ii) We write $\to_e$ for one e-reduction step, that is, a reduction step using the rule $e(x) \to x$. Thus: $C[e(M)] \to_e C[M]$ for a context $C[\ ]$ and a term M in R_e.

(iii) For terms M_1, M_2 of R_e, we write $M_1 \to_f M_2$ ("f" for "frozen") if the redex contracted is not an e-redex nor in the scope of some 'e.' So if $C[e(P)] \to_f^S N$ where S is the contracted redex, then it is not the case that $S \subseteq e(P)$.

Notation 2.2

(i) For notational ease we will henceforth write $\underline{M}$ instead of $e(M)$ and $\underline{R}$ instead of R_e. Terms from $\underline{R}$ are "underlined" terms (even if they contain no actual underlining).

(ii) We write $\twoheadrightarrow$ for the transitive-reflexive closure of $\twoheadrightarrow_f \cup \twoheadrightarrow_e$. (This is, in fact, an ambiguous use of $\twoheadrightarrow$, since it was already in use for not underlined terms. But the present extension of the old $\twoheadrightarrow$ to the case of underlined terms will cause no confusion.)

(iii) In the sequel, $C[\underline{P}_1, \ldots, \underline{P}_p]$ denotes a term such that all underlinings are displayed, that is, $C[P_1, \ldots, P_p]$ contains no underlined subterm.

Example 2.3

(i) Let R be the TRS as in the introduction of this section. Then, the $\underline{R}$-term $F(\underline{H(C)}, G(\underline{C}))$ (in the e-notation: $F(e(H(C)), G(e(C)))$ has the reduction:

$$F\big(\underline{H(C)}, G(\underline{C})\big) \to_e F(H(C), G(\underline{C})) \to_f F(C, G(\underline{C})) \to_f G(\underline{C}) \to_f \underline{C} \to_e C.$$

(ii) Note that the terms $\underline{F(H(C), G(C))}$ and $F(\underline{H(C)}, \underline{G(C)})$ are normal forms with respect to $\to_f$ (f-normal forms).

PROPOSITION 2.4. *Let R be a confluent and left-linear TRS. Then:*

(i) *the reduction $\to_f$ is confluent in $\underline{R}$ (See diagram in figure 2(a).)*

(ii) *the reductions $\to_e$ and $\to_f$ commute in $\underline{R}$. (See diagram in figure 2(b).)*

PROOF. Before starting the proof, note that the shaded arrows have the usual existential meaning. For example, the diagram in Figure 2(a) expresses:

$$\forall M_1, M_2, M_3\ \exists M_4 (M_1 \twoheadrightarrow_f M_2 \,\&\, M_1 \twoheadrightarrow_f M_3 \Rightarrow M_2 \twoheadrightarrow_f M_4 \,\&\, M_3 \twoheadrightarrow_f M_4).$$

(i) Consider in M_1 the maximal underlined subterms. Here "maximal" refers to the subterm ordering $\subseteq$. Replace these subterms by mutually different new variables, in order to "code" these subterms. Do this everywhere in

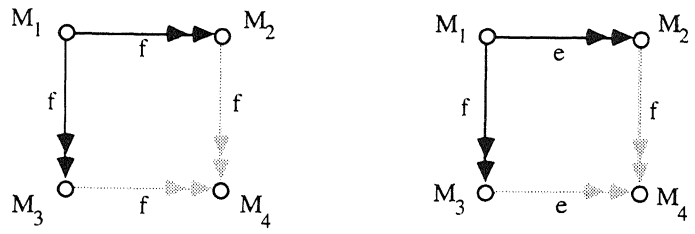


FIGURE 2

the reductions $M_1 \twoheadrightarrow_f M_i$, $i = 2,3$. The resulting reductions $M_1^* \twoheadrightarrow M_i^*$, $i = 2,3$, are "ordinary" (not underlined) reductions in R. Take the common reduct M_4^* according to R; and replace in $M_i^* \twoheadrightarrow M_4^*$ $(i = 2,3)$ the coding variables by the original underlined subterms.

(ii) It suffices to prove the statement for the case that $M_1 \twoheadrightarrow_e M_2$ is one step $M_1 \to_e M_2$. Let this step be in fact $M_1 \equiv C[\underline{N}] \to_e C[N] \equiv M_2$. Then $M_3 \equiv C'[\underline{N}, \ldots, \underline{N}]$ where all descendants of $\underline{N}$ are displayed. Now take $M_4 \equiv C'[N, \ldots, N]$. (Note that the reduction $M_2 \twoheadrightarrow_f M_4$ is possible by left-linearity of R.) □

We will be especially interested in reductions of the form $M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$ where $\underline{P}$ is the only underlined subterm in $C[\underline{P}]$. (Here, and in the sequel, we will permit ourselves a slight abuse of notation by using "$M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$" for "$M \equiv C[P]$ and $C[\underline{P}] \twoheadrightarrow \underline{P}$.") Graphically, the existence of such a reduction is indicated by an arrow as in Figure 3 (cf. the arrows in Figure 1(b). Indeed the two arrows there correspond with the $\to_f$-reductions:

$$M \equiv F(H(C), G(\underline{C})) \to_f F(C, G(\underline{C})) \to_f G(\underline{C}) \to_f \underline{C}$$

$$M \equiv F(H(\underline{C}), G(C)) \to_f F(H(\underline{C}), C) \to_f H(\underline{C}) \to_f \underline{C}).$$

In the situation of Figure 3, we will sometimes say that (the displayed occurrence of) P can be "pulled up from M." We will also say that M "collapses to (the displayed occurrence of) P".

Remark 2.5. Since in $C[\underline{P}] \twoheadrightarrow \underline{P}$ the subterm P initially is "frozen," it might be thought that $C[\underline{P}] \twoheadrightarrow \underline{P}$ implies $C[z] \twoheadrightarrow z$ for a fresh variable z. This is not the case as the following example shows: Let R have the reduction rules

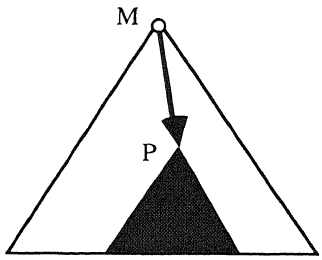
$$F(x) \to G(x, x)$$
$$G(C, x) \to x$$
$$H(x) \to x.$$

Then $F(\underline{H(C)}) \twoheadrightarrow \underline{H(C)}$ in view of the reduction sequence

$$F(\underline{H(C)}) \to G(\underline{H(C)}, \underline{H(C)}) \to G(H(C), \underline{H(C)}) \to G(C, \underline{H(C)}) \to \underline{H(C)}.$$

However, $F(z) \twoheadrightarrow z$ does not hold. The explanation is that in a reduction $C[\underline{P}] \twoheadrightarrow \underline{P}$ not *all* descendants of the initial $\underline{P}$ need to remain frozen; only the $\underline{P}$ on the "main line" of descendants leading to the ultimate $\underline{P}$ in the right-hand side of $C[\underline{P}] \twoheadrightarrow \underline{P}$ must be frozen. As the above reduction sequence

FIGURE 3



shows, some descendants of the initial $\underline{P}$ in $C[\underline{P}]$, not in the main line of descendants, may actually play a necessary role in the collapse to the ultimate $\underline{P}$. (What does hold is the implication $C[\underline{P}] \twoheadrightarrow_f \underline{P} \Rightarrow C[z] \twoheadrightarrow z$ for a fresh variable z. The next proposition (part (i)) generalizes this obvious fact.)

PROPOSITION 2.6. *Let R be a confluent and left-linear TRS. Then*:

(i) $C[\underline{P}] \twoheadrightarrow_f C'[\underline{P}, \ldots, \underline{P}] \Leftrightarrow$
$C[z] \twoheadrightarrow C'[z, \ldots, z]$ *for a fresh variable* $z \Leftrightarrow$
$C[\underline{Q}] \twoheadrightarrow_f C'[\underline{Q}, \ldots, \underline{Q}]$ *for all* Q.

(ii) *Let* $C[\underline{P}] \twoheadrightarrow_f C'[\underline{P}, \ldots, \underline{P}]$ *and* $P \equiv C''[Q]$. *Then* $C[C''[\underline{Q}]] \twoheadrightarrow_f C'[C''[\underline{Q}], \ldots, C''[\underline{Q}]]$.

PROOF. Routine. □

PROPOSITION 2.7. *Let R be a confluent and left-linear TRS. Let* $C[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow^k \underline{P}$ (*i.e., a reduction of k steps of* $\rightarrow_e$ *or* $\rightarrow_f$). *Then for some occurrence of* $\underline{P}$ *in* $C[\underline{P}, \ldots, \underline{P}]$ *and some* $k' \leq k$:

$$C[P, \ldots, \underline{P}, \ldots, P] \twoheadrightarrow^{k'} \underline{P}.$$

PROOF. Consider a reduction $C[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow^k \underline{P}$. Now the final $\underline{P}$ can be traced back to a unique ancestor $\underline{P}$ in $C[\underline{P}, \ldots, \underline{P}]$. Removing the underlining of the other $\underline{P}$ in $C[\underline{P}, \ldots, \underline{P}]$ we obtain $C[P, \ldots, \underline{P}, \ldots, P]$. Clearly, there is now a reduction $C[P, \ldots, \underline{P}, \ldots, P] \twoheadrightarrow \underline{P}$ which is the "same" as the original reduction $C[\underline{P}, \ldots, \underline{P}, \ldots, \underline{P}] \twoheadrightarrow \underline{P}$ except that we possibly gain some e-steps (removals of underlinings). □

LEMMA 2.8. *Let R be a confluent and left-linear TRS. Let* $C[\underline{P}] \twoheadrightarrow \underline{P}$ *and* $Q \twoheadrightarrow P$. *Then* $C[\underline{Q}] \twoheadrightarrow \underline{Q}$.

PROOF. Suppose $C[\underline{P}] \twoheadrightarrow^k \underline{P}$. We will prove the lemma by induction on k. The case $k = 0$ is trivial. Now let

$$C[\underline{P}] \rightarrow C'[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow^{k-1} \underline{P}.$$

By Proposition 2.7, we have for some occurrence of $\underline{P}$ in $C'[\underline{P}, \ldots, \underline{P}]$ and some $k' \leq k - 1$:

$$C'[P, \ldots, \underline{P}, \ldots, P] \twoheadrightarrow^{k'} \underline{P}.$$

By the induction hypothesis, $C'[P, \ldots, \underline{Q}, \ldots, P] \twoheadrightarrow \underline{Q}$. So we have

$$\begin{aligned} C[\underline{Q}] &\rightarrow C'[\underline{Q}, \ldots, \underline{Q}, \ldots, \underline{Q}] \twoheadrightarrow C'[Q, \ldots, \underline{Q}, \ldots, Q] \\ &\twoheadrightarrow C'[P, \ldots, \underline{Q}, \ldots, P] \twoheadrightarrow \underline{Q}. \quad \square \end{aligned}$$

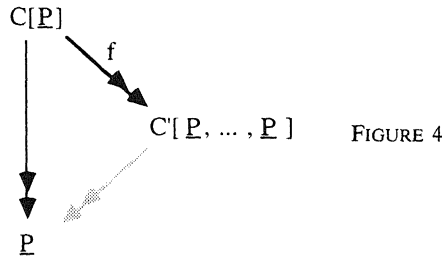


FIGURE 4

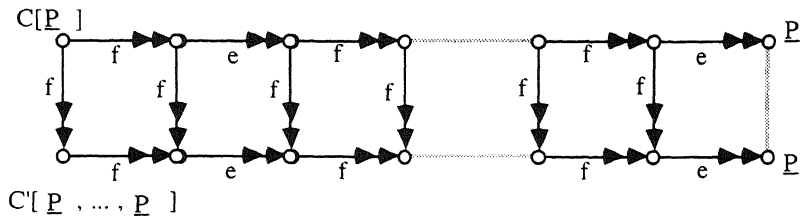


FIGURE 5

PROPOSITION 2.9. *Let R be a confluent and left-linear TRS. Let* $C[\underline{P}] \twoheadrightarrow \underline{P}$ *and let* $C[\underline{P}] \twoheadrightarrow_f C'[\underline{P}, \ldots, \underline{P}]$, *where all occurrences of* $\underline{P}$ *in* $C'[\underline{P}, \ldots, \underline{P}]$ *are displayed.*

Then $C'[\underline{P}, \ldots, \underline{P}]$ *contains at least one occurrence of* $\underline{P}$ *and* $C'[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow \underline{P}$ (*see Figure* 4).

PROOF. That $C'[\underline{P}, \ldots, \underline{P}]$ contains some occurrence of $\underline{P}$ follows immediately from $C'[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow \underline{P}$, since underlinings cannot be created during a reduction.

The proof of $C'[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow \underline{P}$ follows from the diagram in Figure 5. Note that the given reduction $C[\underline{P}] \twoheadrightarrow \underline{P}$ consists of some sequence of $\twoheadrightarrow_f$ and $\twoheadrightarrow_e$ reductions; it is displayed in the upper part of the diagram in Figure 5.

This diagram construction is possible by Proposition 2.4. Note that the right-hand side of the diagram is the *empty* reduction $\underline{P} \twoheadrightarrow_f \underline{P}$ (i.e., consisting of zero steps), since $\underline{P}$ is an f-normal form. Hence, the lower side of the reduction diagram gives us a reduction $C'[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow \underline{P}$. □

LEMMA 2.10. *Let R be a confluent and left-linear TRS.*
If $C[\underline{P}] \twoheadrightarrow \underline{P}$ *and* $P \twoheadrightarrow Q$, *then* $C[\underline{Q}] \twoheadrightarrow \underline{Q}$.

PROOF. Suppose $C[\underline{P}] \twoheadrightarrow^k \underline{P}$. We will prove the proposition by induction on k. The case $k = 0$ is trivial: then $C[\underline{P}] \equiv \underline{P}$ and indeed $\underline{Q} \twoheadrightarrow Q$. Induction hypothesis: the statement holds for $k - 1$ ($k > 0$). Now let $C[\underline{P}] \twoheadrightarrow {}^k\underline{P}$. So $C[\underline{P}] \to C'[\underline{P}, \ldots, \underline{P}] \twoheadrightarrow^{k-1} \underline{P}$. By Proposition 2.7, we have a reduction $C'[P, \ldots, \underline{P}, \ldots, P] \twoheadrightarrow^{k'} \underline{P}$ for some $k' \leq k - 1$ and for some occurrence of P. Hence, by the induction hypothesis, $C'[P, \ldots, \underline{Q}, \ldots, P] \twoheadrightarrow \underline{Q}$.

By Proposition 2.9, since $C'[P, \ldots, \underline{Q}, \ldots, P] \twoheadrightarrow_f C'[Q, \ldots, \underline{Q}, \ldots, Q]$, we have $C'[Q, \ldots, \underline{Q}, \ldots, Q] \twoheadrightarrow Q$. Concatenating this reduction with $C[\underline{Q}] \to C'[\underline{Q}, \ldots, \underline{Q}, \ldots, \underline{Q}] \twoheadrightarrow C'[Q, \ldots, \underline{Q}, \ldots, Q]$, we have indeed $C[\underline{Q}] \twoheadrightarrow \underline{Q}$. □

Remark 2.11. From the preceding propositions we see that the relation $C[\underline{P}] \twoheadrightarrow \underline{P}$ is preserved under convertibility ($=$, the equivalence generated by $\twoheadrightarrow$, that is, by $\to_e$, $\to_f$.). For, combining Lemma's 2.8 and 2.10, we have:

$$C[\underline{P}] \twoheadrightarrow \underline{P} \quad \& \quad P = Q \Rightarrow C[\underline{Q}] \twoheadrightarrow \underline{Q}.$$

Moreover, $C[\underline{P}] \twoheadrightarrow \underline{P}$ is preserved under any reduction of $C[\underline{P}]$ which leaves $\underline{P}$ unaffected, as Proposition 2.9 states ($\underline{P}$ may be multiplied, though.)

3. *Mixed Terms*

We will now consider disjoint unions, or as we will call them, *direct sums* $R_b \oplus R_w$ of TRSs R_b, R_w with disjoint alphabets. Henceforth, we assume that R_b, R_w are left-linear and complete. Let $\mathscr{F}$ be a set of function and constant symbols, and let $\mathscr{V}$ be a countably infinite set of variables. Then $\mathrm{Ter}(\mathscr{F}, \mathscr{V})$ is the set of terms constructed from $\mathscr{F}$ and $\mathscr{V}$. If R_i ($i = \mathrm{b}, \mathrm{w}$) are TRSs with rule sets $\mathrm{Red}(R_i)$, terms $\mathrm{Ter}(\mathscr{F}_i, \mathscr{V})$ such that $\mathscr{F}_b$ and $\mathscr{F}_w$ are disjoint, then $R_b \oplus R_w$ is the TRS with terms $\mathrm{Ter}(\mathscr{F}_b \cup \mathscr{F}_w, \mathscr{V})$ and reduction rules $\mathrm{Red}(R_b) \cup \mathrm{Red}(R_w)$. Instead of $\mathrm{Ter}(\mathscr{F}_b \cup \mathscr{F}_w, \mathscr{V})$ we will also write $\mathrm{Ter}(R_b \oplus R_w)$.

As a mnemonic device, we will call the function and constant symbols of R_b *black* and those of R_w *white*. To distinguish in print between them, the black symbols are capitals and the white symbols are lower case. Thus, a term $M \in \mathrm{Ter}(R_b \oplus R_w)$, in its tree notation, is a constellation of black and white "triangles," as in Figure 6. Here, the root of M is the leading symbol of M.

Note that if R_b and R_w are complete (as always assumed in this paper), then every term in $\mathrm{Ter}(R_b \oplus R_w)$ has a normal form. This can be easily proved using innermost reductions (in which the only redexes reduced are those containing no proper subredexes). Moreover, the normal form is unique, since $R_b \oplus R_w$ is confluent (by the main theorem in Toyama [1987a]). The normal form of term t will be denoted by $t\downarrow$.

Definition 3.1

(i) Let $M \equiv C[B_1, \ldots, B_n] \in \mathrm{Ter}(R_b \oplus R_w)$ and $C[\ ] \neq \Box$. Then we write $M \equiv C[\![B_1, \ldots, B_n]\!]$ if $C[, \ldots,]$ is a context of R_b and $\mathrm{root}(B_i) \in \mathscr{F}_w$ for $i = 1, \ldots, n$. (Likewise with b, w interchanged.) The B_i are called the *principal* subterms of M.

(ii) The set $\mathbf{S}(M)$ of *special* subterms (more precisely, subterm occurrences) is inductively defined as follows:

$$\mathbf{S}(M) = \begin{cases} \{M\} & \text{if} \quad M \in \mathrm{Ter}(R_d)\ (d = \mathrm{b}, \mathrm{w}) \\ \{M\} \cup \bigcup_i \mathbf{S}(B_i) & \text{if} \quad M \equiv C[\![B_1, \ldots, B_n]\!]\ (n > 0), \end{cases}$$

(iii) $\mathbf{S}_d(M) = \{N \mid N \in \mathbf{S}(M) \quad \text{and} \quad \mathrm{root}(N) \in \mathscr{F}_d\}(d = \mathrm{b}, \mathrm{w})$.
(iv) $\mathbf{G}_d(M) = \{N \mid M \twoheadrightarrow N \quad \text{and} \quad \mathrm{root}(N) \in \mathscr{F}_d\}\ (d = \mathrm{b}, \mathrm{w})$.

Definition 3.2. Let $M \in \mathrm{Ter}(R_b \oplus R_w)$. Then:

$$\mathrm{rank}(M) = \begin{cases} 1 & \text{if} \quad M \in \mathrm{Ter}(R_d)\ (d = \mathrm{b}, \mathrm{w}) \\ \max_i\{\mathrm{rank}(B_i)\} + 1 & \text{if} \quad M \equiv C[\![B_1, \ldots, B_n]\!]\ (n > 0) \end{cases}$$

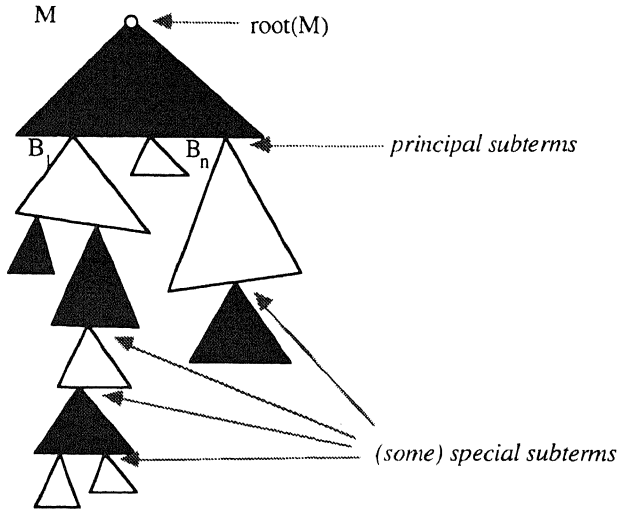


FIGURE 6

The following fact (where $\rightarrow$ is reduction in $R_b \oplus R_w$) has a routine proof that is omitted.

PROPOSITION 3.3. *If $M \rightarrow N$, then $rank(M) \geq rank(N)$.*

PROPOSITION 3.4. *Let $M \twoheadrightarrow N$ where both M, N have a black root. Then there exists a reduction $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \equiv N$ such that all M_i $(i = 0, \ldots, n)$ have a black root.*

PROOF. Let $M \twoheadrightarrow^k N$ $(k \geq 0)$. We will prove the proposition by induction on k. The case $k = 0$ is trivial. Now let $M \rightarrow M' \twoheadrightarrow^{k-1} N$. If the root of M' is black, then we are through, by the induction hypothesis. If the root of M' is white, then there exists a context $C[\ ]$ with black root such that $M \equiv C[M']$ and $C[\ ] \rightarrow \Box$, the trivial context. Thus, we have a reduction $M \equiv C[M'] \twoheadrightarrow C[N] \rightarrow N$ in which all terms have a black root. $\Box$

LEMMA 3.5. *Let $M \rightarrow N$. Let Q be a special subterm of N with white (black) root. Then there is a special subterm P of M with white (black) root such that $P \equiv Q$ or $P \rightarrow Q$.*

PROOF. Since $M \rightarrow N$, there exists a rewriting rule $C_l[x_1, \ldots, x_n] \rightarrow C_r[x_{i1}, \ldots, x_{im}]$, a context $C[\ ]$, and mixed terms $M_1, \ldots, M_n$ such that $M \equiv C[C_l[M_1, \ldots, M_n]]$ and $N \equiv C[C_r[M_{i1}, \ldots, M_{im}]]$ where $M_{i1}, \ldots, M_{im} \in \{M_1, \ldots, M_n\}$.

Case 1. Q and $C_r[M_{i1}, \ldots, M_{im}]$ are disjoint. Q occurs in the context $C[\ ]$ as a special subterm. Thus, we can take Q as P in $M \equiv C[C_l[M_1, \ldots, M_n]]$.

Case 2. $Q \subset C_r[M_{i1}, \ldots, M_{im}]$. As Q is a special and proper subterm occurrence of $C_r[M_{i1}, \ldots, M_{im}]$, Q must occur in some M_{ij}. Since $M_{ij} \in \{M_1, \ldots, M_n\}$, we can take Q as P in M as a special subterm.

Case 3. $Q \equiv C_r[M_{i1}, \ldots, M_{im}]$. If $C_l[M_1, \ldots, M_n]$ has a white root, then take $C_l[M_1, \ldots, M_n]$ in M as P. It is clear that $P \rightarrow Q$ and P is a special

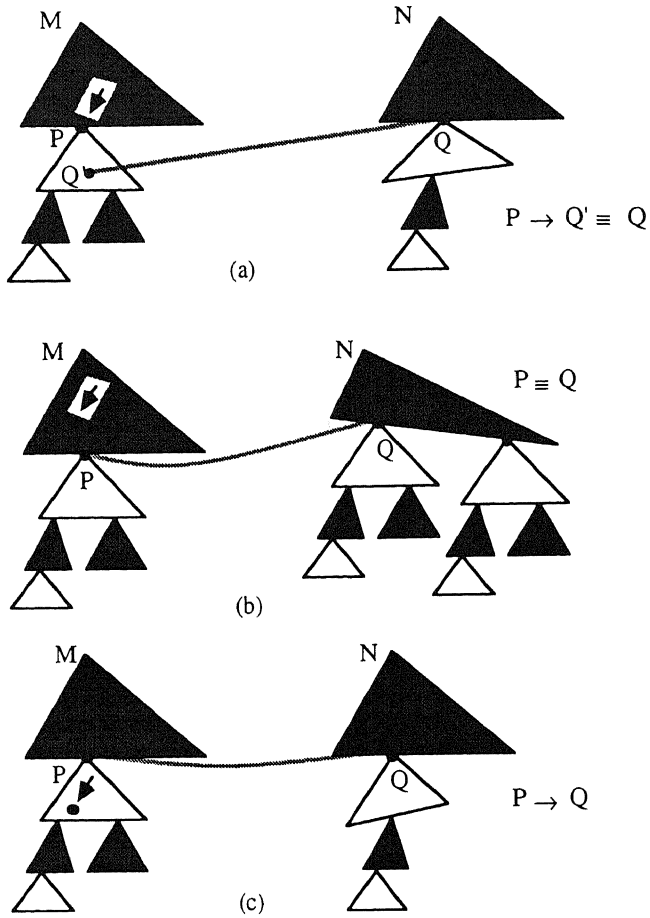


FIGURE 7

subterm with white root (see Figure 7(a)). If $C_l[M_1, \ldots, M_n]$ has a black root then $C_r[, \ldots,] \equiv \square$ and Q must be in $\{M_1, \ldots, M_n\}$, say $Q \equiv M_p$. Thus, we can take M_p in M as a special subterm (see Figure 7(b)).

Case 4. $C_r[M_{i1}, \ldots, M_{im}] \subset Q$. Let $Q \equiv C_a[C_r[M_{i1}, \ldots, M_{im}]]$ where $C_a[, \ldots,] \neq \square$. Then there exists a context $C_b[\ ]$ such that $N \equiv C_b[Q] \equiv C_b[C_a[C_r[M_{i1}, \ldots, M_{im}]]]$ and $M \equiv C_b[C_a[C_l[M_1, \ldots, M_n]]]$. Take $P \equiv C_a[C_l[M_1, \ldots, M_n]]$ in M. Then, $P \to Q$ and P is a special subterm with white root of M (see Figure 7(c)). $\square$

LEMMA 3.6. *Let M have a black root ($\in \mathbf{F}_b$) and suppose $M \twoheadrightarrow N$ where N has a white root. Then M has a special subterm P with white root such that $M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$ and $P \twoheadrightarrow N$. (See Figure 8.)*

PROOF. Suppose $M \twoheadrightarrow^k N$. We will prove the proposition by induction on k. The case $k = 1$ is trivial; then N must be in fact one of the principal subterms M_r of $M \equiv C'[\![M_1, \ldots, M_r, \ldots, M_n]\!]$ and we can take $P \equiv M_r$.

Induction Hypothesis: Suppose the statement is proved for $k - 1$. Now consider $M \twoheadrightarrow^k N$, that is, $M \to M' \twoheadrightarrow^{k-1} N$ for some M'.

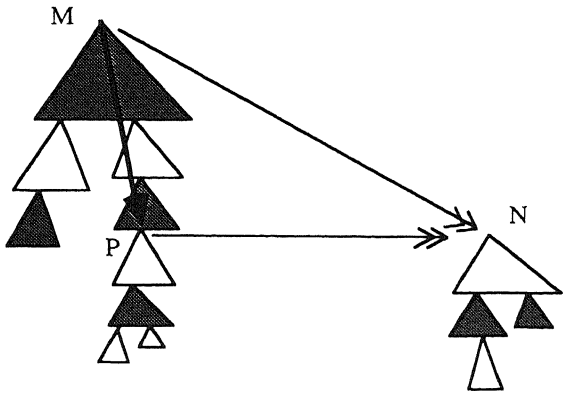


FIGURE 8

Case 1. The root of M' is white. Then $M \equiv C'[\![M_1, \ldots, M_r, \ldots, M_n]\!] \to M' \equiv M_r$ for some r. Take $P \equiv M_r$.

Case 2. The root of M' is black. According to the induction hypothesis M' has a special subterm P' with white root such that $M' \equiv C[\underline{P'}] \twoheadrightarrow \underline{P'}$ and $P' \twoheadrightarrow N$. By Lemma 3.5, there is a special subterm $P \in \mathbf{S}_w(M)$ such that $P \to P'$ or $P \equiv P'$. We distinguish two subcases:

Case 2.1. $P \to P'$. Then $M \equiv C[P] \to M' \equiv C[P']$. By Lemma 2.8, $M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$. Since $P \to P' \twoheadrightarrow N$ the statement is proved for this case.

Case 2.2. $P \equiv P'$. Then $M \equiv C'[\underline{P}] \to C^*[\underline{P}, \ldots, \underline{P}, \ldots, \underline{P}] \twoheadrightarrow_e C^*[P, \ldots, \underline{P}, \ldots, P] \equiv M' \equiv C[\underline{P}] \equiv C[\underline{P'}] \twoheadrightarrow \underline{P'} \equiv \underline{P}$. □

3.7. ESSENTIAL SUBTERMS. As the last lemma (3.6) states, if M has a black root all reductions of M to a term with white root can be "factored through" reductions of M to its special subterms with white root. Of these special subterms with white root, some are even more special: the *essential* subterms of M. As we will see, every collapse reduction of M to a special subterm Q with white root can be factored as a collapse of M to an essential subterm P followed by a collapse of P to Q. (See Figure 9.)

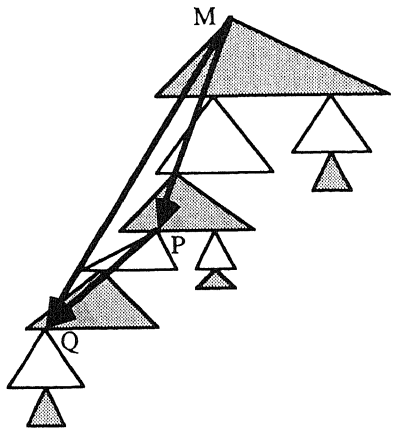
Definition 3.7.1. Let M have a black root. Let P be a special subterm of M with white root such that M collapses to P. Then P is an *essential* subterm (occurrence) of M if there is no special subterm P' with white root such that $P \not\equiv P'$, M collapses to P', and P' collapses to P. The set of essential subterms of M is $\mathbf{E}(M)$. (Likewise, with colors interchanged.)

In other words: Let $\text{root}(M) \in \mathscr{F}_b$. Then the essential subterms of M are the *maximal* elements in the set $\{N \in \mathbf{S}_w(M) \mid M \text{ collapses to } N\}$, partially ordered by the relation "...collapses to...".

LEMMA 3.7.2. *Let M have a black root, and suppose $M \twoheadrightarrow N$ where N has a white root. Then for some essential subterm P of M: $P \twoheadrightarrow N$.*

PROOF. Immediately by Lemma 3.6 and Definition 3.7.1. □

FIGURE 9



3.8. DETERMINISTIC TERMS. In the preceding subsection, we have already set up some notions to discuss the "collapsing behavior" of mixed terms. We will now introduce an important property of this collapsing behavior—first for the case of a single TRS.

Definition 3.8.1. Let R be a TRS and $M \in \mathrm{Ter}(R)$. Then M is a *nondeterministic* term if

(i) $M \equiv C[P, Q]$ and $C[\underline{P}, Q] \twoheadrightarrow \underline{P}, C[P, \underline{Q}] \twoheadrightarrow \underline{Q}$ (Figure 10(a)), *or*
(ii) $M \equiv C[P]$, $P \equiv C'[Q], C[\underline{P}] \twoheadrightarrow \underline{P}$, $C[\overline{C'}[\underline{Q}]] \twoheadrightarrow \underline{Q}$ but not $C'[\underline{Q}] \twoheadrightarrow \underline{Q}$ (Figure 10(b)); here $P \nrightarrow Q$ denotes that P cannot collapse to Q).

An example of a nondeterministic term was given in the introduction of Section 2, for nondeterminism of type (i). As an example of nondeterminism of type (ii) consider $R = \{F(x) \to G(x, x), G(D, x) \to x, G(H(y), D) \to y, H(D) \to D, C \to D\}$. This TRS is left-linear and complete. Now take $M \equiv F(H(C))$; then $F(\underline{H(C)}) \twoheadrightarrow \underline{H(C)}$, $F(H(\underline{C})) \twoheadrightarrow \underline{C}$, but not $H(\underline{C}) \twoheadrightarrow \underline{C}$.

Remark 3.8.2. The phenomenon of nondeterministic terms is caused by ambiguities between the rewrite rules (i.e., the presence of "critical pairs"). Indeed, one can prove: In an orthogonal TRS (i.e., left-linear, non-ambiguous TRS) all terms are deterministic. The proof is rather lengthy and, since we have no need for this fact here, is not included in this paper.

Definition 3.8.3. Let R_b, R_w be arbitrary TRSs and let $M \in \mathrm{Ter}(R_b \oplus R_w)$. Then M is a *mixed nondeterministic term* if M has at least two essential subterm occurrences. (See Figure 11.)

Remark 3.8.4. There are R_b, R_w and terms M, M' with $M \to M'$ such that M' is mixed nondeterministic, but M is not. Example: consider $R_b = \{G(x) \to F(x, x), F(x, C) \to x, F(C, x) \to x\}$, $R_w = \{g(x) \to x\}$ and $M \equiv G(g(C)) \to F(g(C), g(C)) \equiv M'$.

Clearly, a mixed nondeterministic term is nondeterministic in the sense of Definition 3.8.1. In the sequel, we will say that a term M *has color change* if $\mathrm{root}(M)$ is black and $\mathrm{root}(M\downarrow)$ is white, or vice versa. The following lemma plays an important role in Section 5.

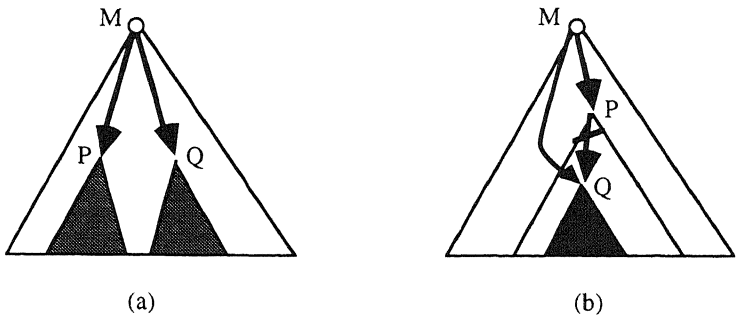


FIGURE 10

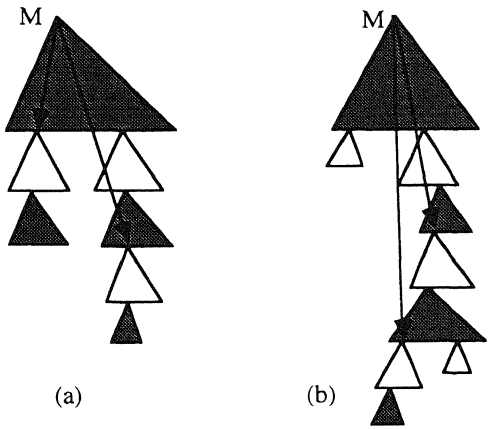


FIGURE 11

LEMMA 3.8.5. (MAIN LEMMA). *Let R_b, R_w be left-linear and complete. Let M be terminating and let M have color change. Then M has exactly one essential subterm.*

PROOF. See Appendix A. □

4. *Termination for the Direct Sum*

In this section, we will prove the main result, that is, the termination property for the direct sum $R_b \oplus R_w$ of left-linear and complete R_b, R_w. To this end, we define for a term $M \in \mathrm{Ter}(R_b \oplus R_w)$ two terms: the *black projection* $M^b \in \mathrm{Ter}(R_b)$ of M, and the *white projection* $M^w \in \mathrm{Ter}(R_w)$ of M. Roughly, the black/white projections of M contain precisely the "information" in the black, respectively white, part of M. In fact, we will prove that if M is a supposed minimal (with respect to length) term with white root, admitting an infinite reduction, then the white projection M^w has already an infinite reduction. As M^w is in $\mathrm{Ter}(R_w)$, this is in contradiction with the termination property of R_w and we will have proved termination for $R_b \oplus R_w$.

VARIABLE CONVENTION 4.1. *From now on we will assume that every term $M \in Ter(R_b \oplus R_w)$ has only "x" as variable occurrences*, unless other variables are explicitly displayed. Since $R_b \oplus R_w$ is left-linear, this variable convention may be assumed in the sequel without loss of generality.

While the definition and properties of the white projection M^{w} (the black projection M^{b}) of M will be given and proved in Section 5, it is sufficient for the rest of this section to assume the properties as stated in the following proposition.

PROPOSITION 4.2. *Let every maximal special subterm with black (white) root of $M \in Ter(R_b \oplus R_w)$ be terminating (if it exists). Then the white projection M^w (the black projection M^b) of M has the properties (where the black projection M^b has the properties obtained by interchanging w, b and white and black.):*

(i) *If $M \equiv x$, then $M^w \equiv x$.*
(ii) *If $M \equiv f(M_1, \ldots, M_n)(n \geq 0)$ has a white root, then $M^w \equiv f(M_1^w, \ldots, M_n^w)$.*
(iii) *If M has a black root and an essential subterm N, then $M^w \twoheadrightarrow N^w$.*
(iv) *If $M \twoheadrightarrow x$, then $M^w \twoheadrightarrow x$.*

PROOF. See the next section. □

Remark 4.3. Let $M \equiv C[M_1, \ldots, M_n]$ where $C[, \ldots,]$ is a context in $\mathrm{Ter}(R_{\mathrm{w}})$. Then, from Proposition 4.2(i) and (ii), we have $M^{\mathrm{w}} \equiv C[M_1^{\mathrm{w}}, \ldots, M_n^{\mathrm{w}}]$. From this it follows that $M^{\mathrm{w}} \equiv M$ if $M \in \mathrm{Ter}(R_{\mathrm{w}})$.

Remark 4.4. It is clear that the assumption "*Every maximal special subterm with black root of $M \in Ter(R_b \oplus R_w)$ is terminating (if it exists)*" means that (i) if M has a white root, every principal subterm (if it exists) is terminating and (ii) if M has a black root, M itself is terminating.

Remark 4.5. Let every maximal special subterm with black root of $M \in \mathrm{Ter}(R_{\mathrm{b}} \oplus R_{\mathrm{w}})$ be terminating, and let $M \to N$. Then, it is trivial that every maximal special subterm with black root of N is also terminating. Thus, if M has the white (black) projection by Proposition 4.2, then so has N.

Notation 4.6

(i) We write $M \equiv_{\mathrm{o}} N$ when M, N have the same outermost-layer context, that is, $M \equiv C[\![M_1, \ldots, M_m]\!]$ and $N \equiv C[\![N_1, \ldots, N_m]\!]$ for some M_i, N_i $(i = 1, \ldots, m)$.
(ii) Let $M \equiv C[\![M_1, \ldots, M_m]\!]$ and suppose $M \to^R N$ (i.e., N is obtained from M by contracting the redex occurrence R). If the redex occurrence R occurs in some M_i, we write $M \to_{\mathrm{i}} N$ ("inner reduction"); otherwise, we write $M \to_{\mathrm{o}} N$ ("outer reduction").

Note that $M_1 \twoheadrightarrow_{\mathrm{i}} M_2$, $M_2 \twoheadrightarrow_{\mathrm{i}} M_3$ implies $M_1 \twoheadrightarrow_{\mathrm{i}} M_3$.

PROPOSITION 4.7. *Let $M \to_o N$ where M, N have white roots. Suppose $M \equiv_o A$ and $A \twoheadrightarrow_i M$ (internal reduction), where every maximal special subterm with black root of A is terminating. Then there exists a term B such that $N \equiv_o B$, $A \to_o B$, $B \twoheadrightarrow_i N$ and $A^w \to B^w$. (See diagram in Figure 12.)*

PROOF. Let $A \equiv C[\![A_1, \ldots, A_m]\!]$, $M \equiv C[\![M_1, \ldots, M_m]\!]$ and $N \equiv C'[\![M_{i1}, \ldots, M_{in}]\!](i_j \in \{1, \ldots, m\})$. Take $B \equiv C'[\![A_{i1}, \ldots, A_{in}]\!]$. Then $A \to_{\mathrm{o}} B$ and $B \twoheadrightarrow_{\mathrm{i}} N$. From $A^{\mathrm{w}} \equiv C[A_1^{\mathrm{w}}, \ldots, A_m^{\mathrm{w}}]$ and $B^{\mathrm{w}} \equiv C'[A_{i1}^{\mathrm{w}}, \ldots, A_{in}^{\mathrm{w}}]$, it follows that $A^{\mathrm{w}} \to B^{\mathrm{w}}$. □

PROPOSITION 4.8. *Let $M \twoheadrightarrow N$ where root(N) is white and every maximal special subterm with black root of M is terminating. Then there exists a term A such that $N \equiv_o A$, $A \twoheadrightarrow_i N$, $M \twoheadrightarrow A$, and $M^w \twoheadrightarrow A^w$. (See diagram in Figure 13.)*

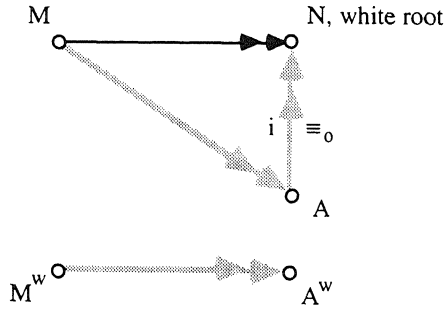


FIGURE 12

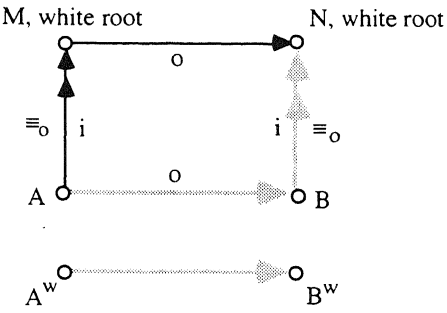


FIGURE 13

PROOF. We will prove the proposition by induction on rank(M).

Basis: rank(M) = 1. This case is trivial: Take $A \equiv N$.

Induction hypothesis: The proposition holds for M with rank(M) < k. Now let M have rank k.

CLAIM. *The proposition holds if* $M \twoheadrightarrow_i N$.

PROOF OF THE CLAIM. Let $M \equiv C[\![M_1, \ldots, M_m]\!] \twoheadrightarrow_i N \equiv C[N_1, \ldots, N_m]$ where $M_i \twoheadrightarrow N_i$ for $i = 1, \ldots, m$. Without loss of generality we may assume that $N_1 \equiv x, \ldots, N_{p-1} \equiv x$, root($N_i$) is white for $p \le i < q$, and root(N_j) is black for $q \le j \le m$. Thus,

$$N \equiv C[x, \ldots, x, N_p, \ldots, N_{q-1}, N_q, \ldots, N_m].$$

By the induction hypothesis, for every M_i ($p \le i < q$) there is a term A_i such that we have the diagram in figure 14.

Now take $A \equiv C[x, \ldots, x, A_p, \ldots, A_{q-1}, M_q, \ldots, M_m]$. Clearly, $M \twoheadrightarrow A$. Since $A_i \equiv_o N_i$ ($p \le i < q$) and both M_j, N_j ($q \le j \le m$) have black root, we have $A \equiv_o N$. Furthermore, $A \twoheadrightarrow_i N$ since $A_i \twoheadrightarrow_i N_i$ ($p \le i < q$) and by Proposition 3.4 the reductions $M_j \twoheadrightarrow N_j$ ($q \le j \le m$) can be taken such that every term in them has a black root. Now

$$M^w \equiv C\left[M_1^w, \ldots, M_{p-1}^w, M_p^w, \ldots, M_{q-1}^w, M_q^w, \ldots, M_m^w\right],$$

$$A^w \equiv C\left[x, \ldots, x, A_p^w, \ldots, A_{q-1}^w, M_q^w, \ldots, M_m^w\right],$$

(for A^w, see Remark 4.3). By Proposition 4.2(iv) we have $M_i^w \twoheadrightarrow x$ ($1 \le i < p$), since $M_i \twoheadrightarrow x$. We had already $M_i^w \twoheadrightarrow A_i^w$ ($p \le i < q$). Hence, $M^w \twoheadrightarrow A^w$. (See figure 15.) (End of the Claim) □

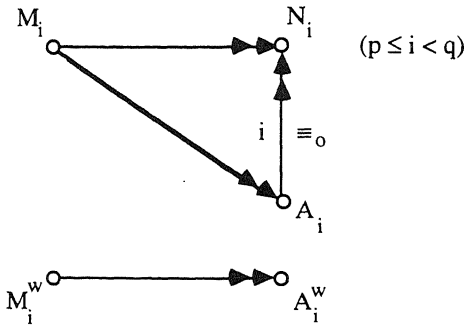


FIGURE 14

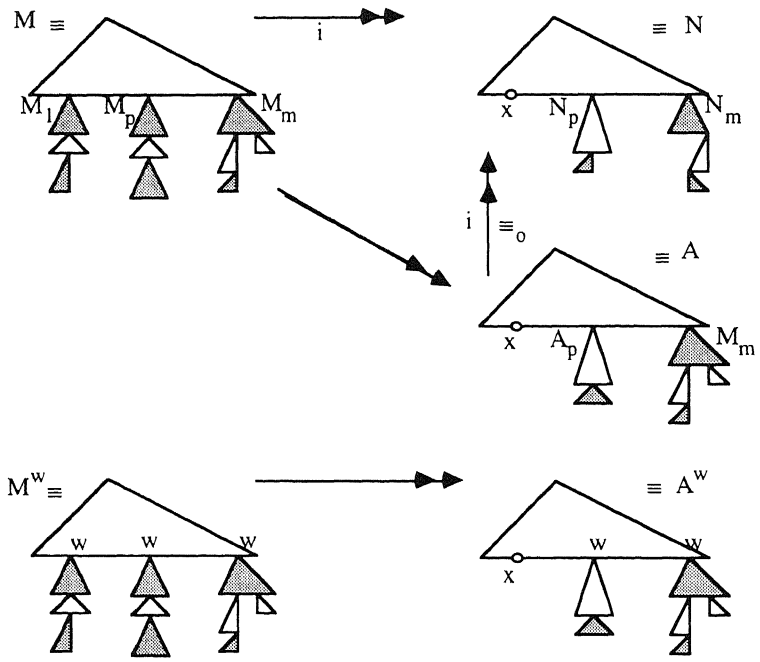


FIGURE 15

Now we will prove the full proposition (without the additional assumption $M \twoheadrightarrow_i N$ as in the Claim) for rank$(M) = k$. We distinguish two cases.

Case 1. The root of M is white.

So M, N have both white roots. Hence, there is, by Proposition 3.4, a reduction $M \twoheadrightarrow N$ in which every term has white root. This reduction can be split into

$$M \twoheadrightarrow_i \to_o \twoheadrightarrow_i \to_o \cdots \twoheadrightarrow_i N.$$

Now we can construct the diagram as in Figure 16.

Here subdiagrams α are justified by the Claim, subdiagrams β by Proposition 4.7 and subdiagrams γ follow by transitivity of $\twoheadrightarrow_i$.

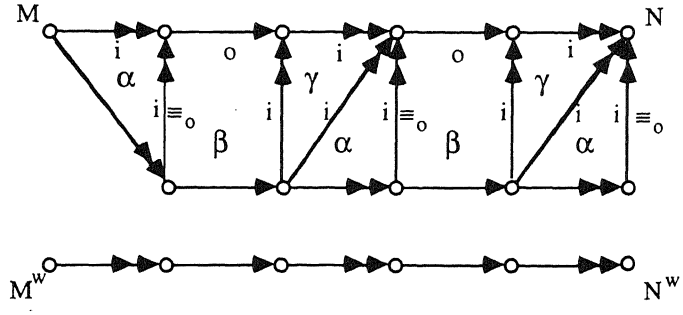


FIGURE 16

Case 2. The root of M is black.

By Lemma 3.7.2, there is an essential subterm Q of M such that $M \twoheadrightarrow Q \twoheadrightarrow N$. By Proposition 4.2(iii), $M^w \twoheadrightarrow Q^w$. Obviously, $\text{rank}(Q) < \text{rank}(M) = k$. Hence, we can construct the diagram in Figure 17, where the triangular subdiagram is obtained by the induction hypothesis applied on Q. □

We are now able to state and prove the main result of our paper:

THEOREM 4.9. *Let R_b, R_w be left-linear and complete. Then $R_b \oplus R_w$ is a terminating TRS.*

PROOF. Let $M \in \text{Ter}(R_b \oplus R_w)$. We will prove by induction on $\text{rank}(M)$ that M does not have an infinite reduction.

Without loss of generality, we may assume that M has a white root. The case $\text{rank}(M) = 1$ is trivial, by assumption. Induction hypothesis: If $\text{rank}(M) < k$, M cannot have an infinite reduction. Now suppose for a proof by contradiction that there is a term M with $\text{rank}(M) = k$ having an infinite reduction $M \equiv M_0 \to M_1 \to M_2 \to \cdots$. Now $\text{rank}(M_0) \geq \text{rank}(M_1) \geq \cdots$; by the induction hypothesis it follows that $\text{rank}(M_0) = \text{rank}(M_1) = \cdots$. Hence, the roots of all M_i are white. Note that every principal subterm of M_i is terminating.

Now infinitely many steps $M_i \to M_{i+1}$ must be in fact $M_i \to_o M_{i+1}$; otherwise, we would have an infinite internal reduction

$$M_k \equiv C_k[\![M_{k,1}, \ldots, M_{k,r}]\!] \to_i \to_i \to_i \cdots,$$

which would yield an infinite reduction of some $M_{k,p}$, in contradiction with the induction hypothesis.

So, we can apply the following diagram construction (Figure 18), using Propositions 4.7 and 4.8 in the same way as for Figure 16. But this means that M^w has already an infinite reduction, in contradiction with the termination property of R_w. □

COROLLARY 4.10. *Let R_b, R_w be left-linear. Then*:

$$R_b \oplus R_w \text{ is complete} \Leftrightarrow R_b \text{ and } R_w \text{ are complete.}$$

PROOF

($\Rightarrow$) is trivial.

($\Leftarrow$) follows from Theorem 4.9 and the theorem in Toyama [1987a] stating that for all TRSs, $R_b \oplus R_w$ is confluent iff R_b, R_w are confluent. □

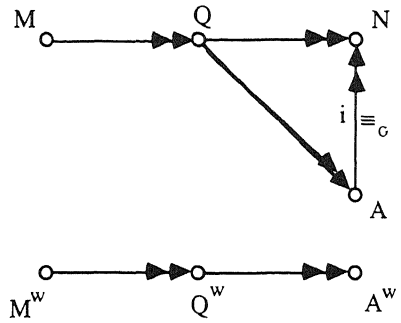


FIGURE 17

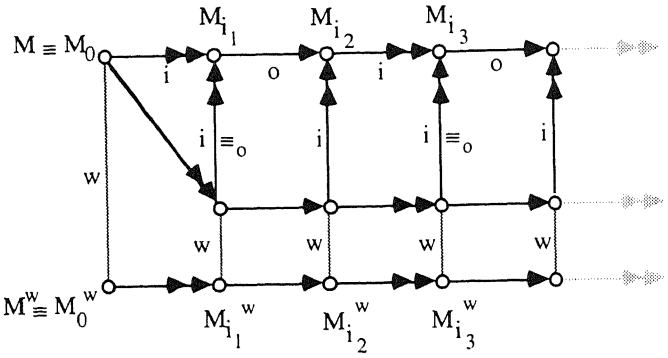


FIGURE 18

5. *Projection of mixed terms*

In this section, we will prove Proposition 4.2 for the direct sum $R_b \oplus R_w$ of left-linear and complete R_b, R_w, that is, the existence of the white (black) projection. To this end, we first define the white (black) projection and next prove that the defined projection satisfies the properties (i)–(iv) in Proposition 4.2.

The definition of the projections is rather subtle and rests heavily upon the Main Lemma 3.8.5. We will prepare the way by an example. Suppose M is structured as in Figure 19(a); a concrete example is: $M \equiv F(g(C), h(C))$ as in Figure 19(b) where $R_b = \{F(x, C) \to x,\ F(C, x) \to x\}$ and $R_w = \{g(x) \to x, h(x) \to x\}$. So $P_1 \equiv g(C)$, $P_2 \equiv h(C)$ are the essential subterms of M. Now suppose we wish to determine the white projection M^w. As M can collapse to P_1 as well as to P_2, the projection M^w should convey the information in both P_1, P_2. The problem is that these subterms are disjoint (in this case). Yet, there is a way to combine them into one term: namely by *piling* them with result as in Figure 19(c), respectively 19(d). Throughout this section, the variable x will play a special role.

Of course, we were lucky in this example, since the white top triangles of P_1, P_2 which we wanted to pile, were indeed "pileable." In the situation of Figure 20, where P_1 is supposed to be again nondeterministic, the piling would not have succeeded, because triangles 1, 2 can be taken such that they cannot be piled. However, our Main Lemma 3.8.5 says that such a situation does not exist and, therefore, piling succeeds as will be proved in more detail below.

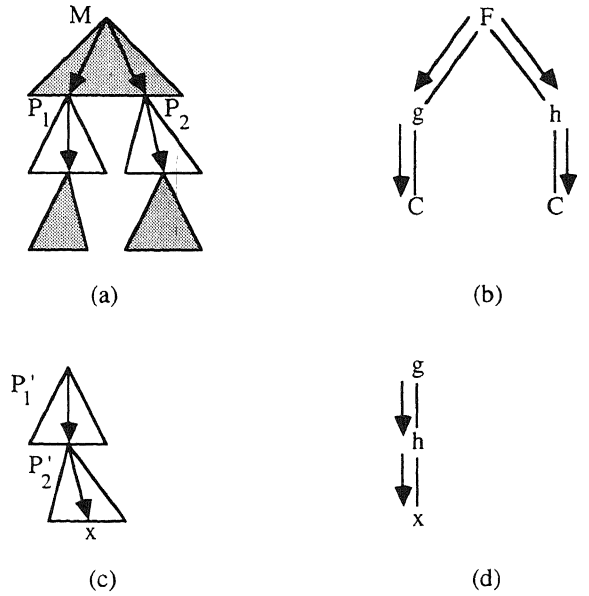


FIGURE 19

Definition 5.1. Let R be a confluent and left-linear TRS. Let $P_1, \ldots, P_p$ be a sequence of terms of R ($p \geq 2$). Then, the term pile$(P_1, \ldots, P_p)$ is defined as follows:

Case 1. $P_i \twoheadrightarrow x$ for $i = 1, \ldots, p$. So $P_i \equiv C_i[x]$ such that $C_i[\underline{x}] \twoheadrightarrow \underline{x}$ (there may be other occurrences of not underlined x's in $C_i[\underline{x}]$).
Then pile$(P_1, \ldots, P_p) \equiv C_1[C_2[\cdots C_{p-1}[C_p[x]] \cdots]]$.

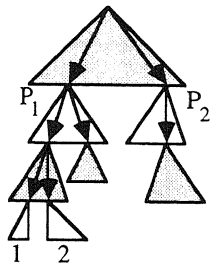
Case 2. Not case 1: Then pile$(P_1, \ldots, P_p)$ is undefined.

Example 5.1.1. Note that pile$(P_1, \ldots, P_p)$ does not merely depend on $P_1, \ldots, P_p$ but also on R. If $R = \{F(x, y) \to x, I(x) \to x\}$ and $P_1 \equiv F(x, x)$, $P_2 \equiv I(x)$, then pile$(P_1, P_2) \equiv F(I(x), x)$. If in R the first rule is replaced by $F(x, y) \to y$, then pile$(P_1, P_2) \equiv F(x, I(x))$.

Remark 5.1.2. The condition in Definition 5.1, that R is confluent and left-linear, is necessary to ensure that pile is a (partial) *function*. Otherwise, taking $R = \{F(x, y) \to x, F(x, y) \to y, I(x) \to x\}$ and $P_1 \equiv F(x, x)$, $P_2 \equiv I(x)$, we would have (see the previous example) pile$(P_1, P_2) \equiv F(x, I(x))$ as well as $F(I(x), x)$. That confluence and left-linearity of R is sufficient to make pile into a function, is easily seen as follows. Now assume that $C[\underline{x}, x] \twoheadrightarrow \underline{x}$ as well as $C[x, \underline{x}] \twoheadrightarrow \underline{x}$. Then this implies (by left-linearity) that $C[x, y] \twoheadrightarrow x$ as well as y, contradicting confluence.

In the sequel, we will use pile for terms of $R_b \oplus R_w$, where R_b, R_w are complete and left-linear. Indeed, the direct sum is then confluent (and, trivially, left-linear), as guaranteed by the theorem in Toyama [1987] stating that the direct sum of confluent TRSs is again confluent. Thus, the operation pile is well defined.

FIGURE 20



Definition 5.2. Let every maximal special subterm with black (white) root of $M \in \mathrm{Ter}(R_0 \oplus R_1)$ be terminating (if it exists). Then, the *white projection* M^{w} of M is inductively defined as follows:

(1) If $M \equiv x$, then $M^{\mathrm{w}} \equiv x$.
(2) $M \equiv f(M_1, \ldots, M_n)$ $(n \geq 0)$ has a white root. Then, $M^{\mathrm{w}} \equiv f(M_1^{\mathrm{w}}, \ldots, M_n^{\mathrm{w}})$.
(3) M has a black root:
  (3.1) M has no essential subterm. Then, $M^{\mathrm{w}} \equiv x$.
  (3.2) M has precisely one essential subterm P. Then, $M^{\mathrm{w}} \equiv P^{\mathrm{w}}$.
  (3.3) M is mixed nondeterministic, with sequence of essential subterms $P_1, \ldots, P_p$. Then, $M^{\mathrm{w}} \equiv \mathrm{pile}(P_1^{\mathrm{w}}, \ldots, P_p^{\mathrm{w}})$.

(The black projection M^{b} is defined by interchanging b, w and black, white.) In case (3.3), the essential subterm occurrences $P_1, \ldots, P_p$ may be ordered by precedence of their head symbol. (The precise ordering is irrelevant.) Note that M^{w} may be undefined, due to the possible undefinedness of $\mathrm{pile}(P_1^{\mathrm{w}}, \ldots, P_p^{\mathrm{w}})$. Lemma 5.7 will however show that in the present situation, where R_{b}, R_{w} are left-linear and complete, if every maximal special subterm with black (white) root of M is terminating then M^{w} (M^{b}) is well defined. Note that (3.2) is not a special case of (3.3) since in general $\mathrm{pile}(N) \not\equiv N$, even if the definition of $\mathrm{pile}(P_1, \ldots, P_p)$ $(p \geq 2)$ is extended to that of $\mathrm{pile}(P_1, \ldots, P_p)$ $(p \geq 1)$. (In fact: $\mathrm{pile}(N) \equiv N \Leftrightarrow \mathrm{pile}(N)$ is defined $\Leftrightarrow N \twoheadrightarrow x$.) Finally, note that in (3.2), (3.3) we have $\mathrm{rank}(P) < \mathrm{rank}(M)$ and $\mathrm{rank}(P_i) < \mathrm{rank}(M)$, respectively.

Example 5.3. (See Figure 21.)

Example 5.4. (See Figure 22.)

Example 5.5. Consider the TRSs $R_{\mathrm{b}} = \{F(C(y), x) \to x, F(x, C(y)) \to x, C(y) \to D\}$, $R_{\mathrm{w}} = \{g(x) \to x, h(x) \to x\}$ with R_{w} containing also a constant 'a'. Then

$$\begin{aligned}(F(g(C(a)), h(C(a))))^{\mathrm{w}} &= \mathrm{pile}\big((g(C(a)))^{\mathrm{w}}, (h(C(a)))^{\mathrm{w}}\big) \\ &= \mathrm{pile}\big(g((C(a))^{\mathrm{w}}), h((C(a))^{\mathrm{w}})\big) \\ &= \mathrm{pile}(g(x), h(x)) = g(h(x)).\end{aligned}$$

Example 5.6. The black projection of the following term (in Figure 23) is undefined; however, by the Main Lemma (3.8.5) such terms cannot exist (when R_{b}, R_{w} are left-linear and complete and every maximal special subterm with white root of M is terminating).

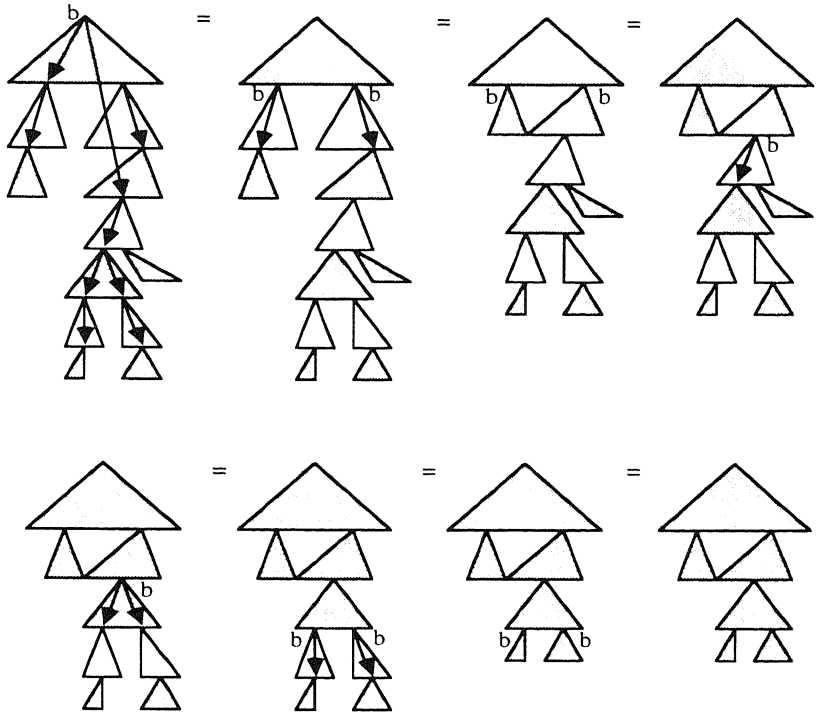


FIGURE 21

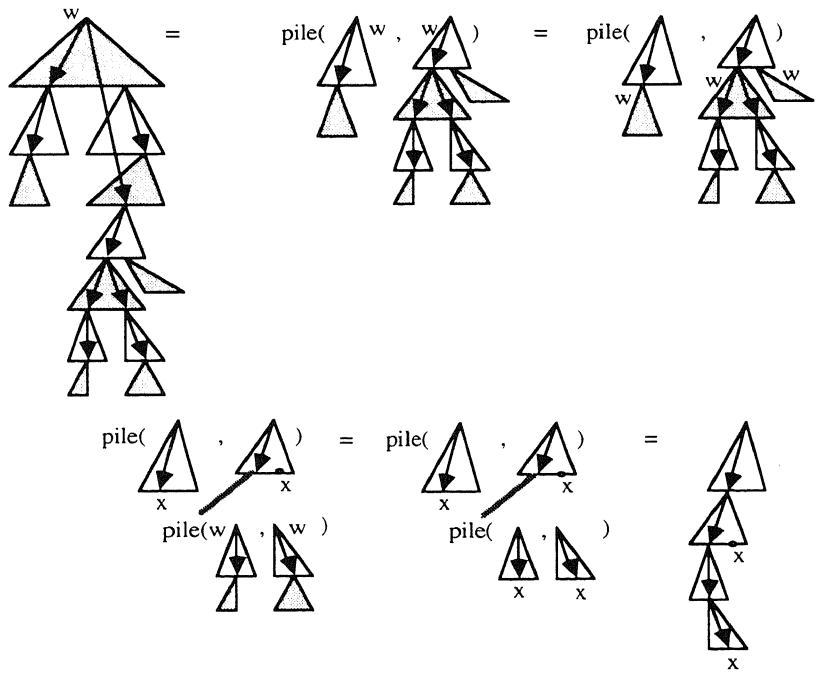


FIGURE 22

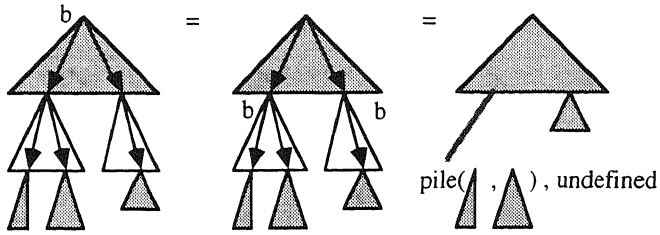


FIGURE 23

In many cases, the result of projecting M to M^{b} or M^{w} will be a term collapsing to the special variable x (i.e. $M^{\mathrm{b}} \twoheadrightarrow x$; respectivey, $M^{\mathrm{w}} \twoheadrightarrow x$.) See for example, Example 5.6. We will now prove simultaneously this fact and the well-definedness of M^{w} (M^{b}) when every maximal special subterm with black (white) root of M is terminating.

LEMMA 5.7. *Let every maximal special subterm with black (white) root of* $M \in Ter(R_b \oplus R_w)$ *be terminating (if it exists). Then the white projection* M^w *(the black projection* M^b*) of* M *is well defined and* $M^w \twoheadrightarrow x \Leftrightarrow root(M\downarrow) \notin \mathscr{F}_w$ *(respectively,* $M^b \twoheadrightarrow x \Leftrightarrow root(M\downarrow) \notin \mathscr{F}_b$*).*

PROOF. We will prove a slightly stronger statement, namely (i) and (ii):

(i) If $\mathrm{root}(M\downarrow) \notin \mathscr{F}_{\mathrm{w}}$, then M^{w} is well-defined and $M^{\mathrm{w}} \twoheadrightarrow x$,
(ii) If $\mathrm{root}(M\downarrow) \in \mathscr{F}_{\mathrm{w}}$ and $M\downarrow \equiv C^*[\![M_1, \ldots, M_m]\!]$ $(m \geq 0)$, then M^{w} is well-defined and $M^{\mathrm{w}}\downarrow \equiv C^*[x, \ldots, x]$. (Hence: Not $M^{\mathrm{w}} \twoheadrightarrow x$.)

We will prove (i) and (ii) by induction on $\mathrm{rank}(M)$.

Basis. $\mathrm{rank}(M) = 1$.

Case 1. $M \in Ter(R_w)$. Then $M^{\mathrm{w}} \equiv M$, by (1) or (2) of Definition 5.2. If $M\downarrow \equiv x$, then $M^{\mathrm{w}} \equiv M \twoheadrightarrow x$, so (i) holds; (ii) holds vacuously. If $\mathrm{root}(M\downarrow) \in \mathscr{F}_{\mathrm{w}}$, then (i) holds vacuously; (ii) holds since $M^{\mathrm{w}}\downarrow \equiv M\downarrow$.

Case 2. $M \in \mathrm{Ter}(R_{\mathrm{b}})$. We may suppose $M \not\equiv x$, since the case $M \equiv x$ was covered in case 1. By (3.1) of Definition 5.2, $M^{\mathrm{w}} \equiv x$. So (i) holds. Statement (ii) holds vacuously.

Induction hypothesis. Assume (i) and (ii) hold for $\mathrm{rank}(M) < k$ $(k \geq 2)$.

Now consider M with $\mathrm{rank}(M) = k$.

Case 1. $root(M) \in \mathscr{F}_w$. Let $M \equiv C[\![M_1, \ldots, M_m]\!] (m \geq 1)$, so $M^{\mathrm{w}} \equiv C[M_1^{\mathrm{w}}, \ldots, M_m^{\mathrm{w}}]$. Without loss of generality, we may assume that $\mathrm{root}(M_i\downarrow) \notin \mathscr{F}_{\mathrm{w}}$ for $1 \leq i < p$ and $\mathrm{root}(M_j\downarrow) \in \mathscr{F}_{\mathrm{w}}$ for $p \leq j \leq m$. So, by the induction hypothesis: every M_i^{w} is well-defined, $M_i^{\mathrm{w}} \twoheadrightarrow x$ $(1 \leq i < p)$, and writing $M_j\downarrow \equiv C_j^*[\![N_{j,1}, \ldots, N_{j,nj}]\!]$ $(n_j \geq 0, p \leq j \leq m)$: $M_j^{\mathrm{w}}\downarrow \equiv C_j^*[x, \ldots, x]$. Thus

$$
\begin{aligned}
M\downarrow &\equiv C[M_1\downarrow, \ldots, M_m\downarrow]\downarrow \\
&\equiv C\left[M_1\downarrow, \ldots, M_{p-1}\downarrow, C_p^*[\![N_{p,1}, \ldots, N_{p,np}]\!], \ldots, C_m^*[\![N_{m,1}, \ldots, N_{m,nm}]\!]\right]\downarrow
\end{aligned}
$$

and

$$M^{\mathrm{w}}\downarrow \equiv C[M_1^{\mathrm{w}}\downarrow,\ldots,M_m^{\mathrm{w}}\downarrow]\downarrow$$
$$\equiv C\big[x,\ldots,x,C_p^*[x,\ldots,x],\ldots,C_m^*[x,\ldots,x]\big]\downarrow.$$

Note that $M_1\downarrow,\ldots,M_{p-1}\downarrow,N_{p,1},\ldots,N_{m,nm}$ are normal forms having roots not in $\mathscr{F}_{\mathrm{w}}$. Therefore, if $\mathrm{root}(M\downarrow)\notin\mathscr{F}_{\mathrm{w}}$, then

$$C\big[x,\ldots,x,C_p^*[x,\ldots,x],\ldots,C_m^*[x,\ldots,x]\big]\downarrow \equiv x$$

and if $\mathrm{root}(M\downarrow)\in\mathscr{F}_{\mathrm{w}}$, then we have a context $C^*[,\ldots,] \equiv C[,\ldots,C_p^*[,\ldots,],\ldots,C_m^*[,\ldots,]]\downarrow$ such that $M\downarrow \equiv C^*[N_1,\ldots,N_n]$ where $N_i \in \{M_1\downarrow,\ldots,M_{p-1}\downarrow,N_{p,1},\ldots,N_{m,nm}\}$ and $M^{\mathrm{w}}\downarrow \equiv C^*[x,\ldots,x]\not\equiv x$ (using $N_{p,1}^{\mathrm{w}}\equiv\cdots\equiv N_{n,nm}^{\mathrm{w}}\equiv x$ by (3.1) of Definition 5.2).

Case 2. $\mathrm{root}(M)\notin\mathscr{F}_{\mathrm{w}}$. Distinguish the subcases:

Case 2.1. M has no essential subterm. Then, $M^{\mathrm{w}}\equiv x$, either by (1) of Definition 5.2 or (3.1). Hence, $M^{\mathrm{w}}\downarrow\equiv x$, and (i) and (ii) hold.

Case 2.2. M has precisely one essential subterm P. Then $M^{\mathrm{w}}\equiv P^{\mathrm{w}}$. Note that $\mathrm{rank}(P)<k$. Thus, by the induction hypothesis every P^{w} is well defined. Since $M\downarrow\equiv P\downarrow$ and $M^{\mathrm{w}}\downarrow\equiv P^{\mathrm{w}}\downarrow$, the claim follows by using the induction hypothesis.

Case 2.3. M has essential subterms $P_1,\ldots,P_p$ $(p>1)$. Note that $\mathrm{rank}(P_i)<k$ for all i. By the Main Lemma 3.8.5, $\mathrm{root}(M\downarrow)\notin\mathscr{F}_{\mathrm{w}}$. Since $M\downarrow\equiv P_i\downarrow$, also $\mathrm{root}(P_i\downarrow)\notin\mathscr{F}_{\mathrm{w}}$ for all i. So, by the induction hypothesis, $P_i^{\mathrm{w}}\twoheadrightarrow x$ for all i. Now $M^{\mathrm{w}}\equiv \mathrm{pile}(P_1^{\mathrm{w}},\ldots,P_p^{\mathrm{w}})$ and since $P_i^{\mathrm{w}}\twoheadrightarrow x$ $(i=1,\ldots,p)$, M^{w} is defined. Obviously, $M^{\mathrm{w}}\equiv\mathrm{pile}(P_1^{\mathrm{w}},\ldots,P_p^{\mathrm{w}})\twoheadrightarrow x$. Hence, (i) is true and (ii) holds vacuously. □

PROPOSITION 5.8. *Let M have a black root and let M be terminating. Suppose P is an essential subterm of M. Then $M^{w}\twoheadrightarrow P^{w}$.*

PROOF. See Definition 5.2 of M^{w}. The only possible cases are (3.2) and (3.3). In case (3.2), $M^{\mathrm{w}}\equiv P^{\mathrm{w}}$. In case (3.3), $M^{\mathrm{w}}\equiv\mathrm{pile}(P_1^{\mathrm{w}},\ldots,P_p^{\mathrm{w}})$ where $P\equiv P_k$ for some $k\in\{1,\ldots,p\}$. From Main Lemma 3.8.5, M cannot have color change, that is, $\mathrm{root}(M\downarrow)$ is black or $M\downarrow\equiv x$. By $M\downarrow\equiv P_1\downarrow\equiv\cdots\equiv P_p\downarrow$ and Lemma 5.7, $P_i^{\mathrm{w}}\twoheadrightarrow x$ $(i=1,\ldots,p)$. Thus, we can write that $P_i^{\mathrm{w}}\equiv C_i[x]$ such that $C_i[\underline{z}]\twoheadrightarrow\underline{z}$. Hence, by definition of "pile":

$$M^{\mathrm{w}}\equiv C_1\big[\cdots\big[C_p[x]\big]\cdots\big],$$

which yields $M^{\mathrm{w}}\twoheadrightarrow C_k[x]\equiv P_k^{\mathrm{w}}$. □

PROPOSITION 5.9 (PROPOSITION 4.2). *Let every maximal special subterm with black (white) root of $M\in Ter(R_b\oplus R_w)$ be terminating (if it exists). Then the white projection M^{w} (the back projection M^{b}) of M has the properties (where the black projection M^{b} has the properties obtained by interchanging w, b and white and black.):*

(i) If $M\equiv x$ then $M^{w}\equiv x$.
(ii) If $M\equiv f(M_1,\ldots,M_n)$ $(n\geq 0)$ has a white root, then $M^{w}\equiv f(M_1^{w},\ldots,M_n^{w})$.
(iii) If M has a black root and an essential subterm N, then $M^{w}\twoheadrightarrow N^{w}$.
(iv) If $M\twoheadrightarrow x$ then $M^{w}\twoheadrightarrow x$.

PROOF. The proposition follows from Definition 5.2, Lemma 5.7, and Proposition 5.8. □

Appendix A

In this appendix, we prove Lemma 3.8.5 (Main Lemma) in Section 3:

LEMMA 3.8.5. MAIN LEMMA. *Let R_b, R_w be left-linear and complete. Let M be terminating and let M have color change. Then M has exactly one essential subterm.*

PROOF. First, we note that from color change of M and Lemma 3.6, M must have at least one essential subterm. Let $\mathbf{P}(M)$ be a predicate that if M is terminating and has color change then M has exactly one essential subterm. We prove $\forall M \mathbf{P}(M)$ by *noetherian induction* (see Huet [1980]), showing the claim:

$$\forall M[\forall N[M \to N \Rightarrow \mathbf{P}(N)] \Rightarrow \mathbf{P}(M)].$$

For a proof of this claim by contradiction, suppose that there exists a terminating term M such that M has color change but more than one essential subterm and every N obtained from M with one step reduction (i.e., $M \to N$) satisfies $\mathbf{P}(N)$. Without loss of generality, we may suppose that M has a black root.

Let N be a term obtained from M with one step reduction. If N has a white root, then we can write $M \equiv C[\![M_1, \ldots, M_k, N, M_{k+1}, \ldots, M_m]\!] \to N$. From Proposition 2.9, N must contain all the essential subterms of M. In particular, N must contain an essential subterm P of M as a proper subterm, since M has more than one essential subterm. But this contradicts the fact that N is also an essential subterm of M. Thus, N must have a black root. From $M\downarrow \equiv N\downarrow$, N has color change. N is clearly terminating. Hence, N must have exactly one essential subterm since $\mathbf{P}(N)$ holds.

Let P and Q be two essential subterms of M. Let M_P (M_Q, respectively) be the minimal special subterm occurrence with black root that contains P (Q respectively). We have to show that all the possible positions of M_P and M_Q in M contradict the uniqueness of the essential subterm of N.

Case 1. M_P and M_Q coincide.

Let $M \equiv C[M_P] (\equiv C[M_Q])$. Since M_P is not only the minimal special subterm occurrence with black root containing P but also that containing Q, P and Q must occur independently in M_P. Thus, we can write $M_P \equiv C_P[P, Q]$.

If $C[z]$ is not in normal form, we have a reduction $C[z] \to C'[z, \ldots, z]$. By Proposition 2.9, $C'[z, \ldots, z]$ must contain at least one occurrence of z. Thus, $N \equiv C'[M_P, \ldots, M_P]$ keeps two essential subterms P and Q; this contradicts the fact that N must have exactly one essential subterm. Hence, $C[z]$ must be a normal form.

Assume that $C_P[z, z']$ is a normal form. Then $C[C_P[P\downarrow, Q\downarrow]]$ is a normal form since $C[C_P[z, z']]$ is a normal form and $P\downarrow$ and $Q\downarrow$ have white roots. However, this contradicts color change of M. Thus, $C_P[z, z']$ must be not in normal form. We can write $C_P[z, z'] \to C'_P[z, \ldots, z, z', \ldots, z']$. By Proposition 2.9, $C'_p[z, \ldots, z, z', \ldots, z']$ must contain the occurrences of z and z'. Thus, we

have $N \equiv C[C'_P[P, \ldots, P, Q, \ldots, Q]]$ in which P and Q are two essential subterms of N: contradiction. Thus, Case 1 is impossible.

Case 2. $M_Q \subset M_P$ (or $M_P \subset M_Q$).

Case 2.1. $M_Q \not\subset P$.

Let $M \equiv C[M_P]$, $M_P \equiv C_P[P, M_Q]$ and $M_Q \equiv C_Q[Q]$. From a similar reason as for Case 1, it follows that $C[z]$ and $C_P[z, z']$ must be normal forms. If $C_Q[z]$ is a normal form, then $C[C_P[P\downarrow, C_Q[Q\downarrow]]]$ must be a normal form of M: contradiction to color change of M. Thus, $C_Q[z]$ is not a normal form. Let $C_Q[z] \to C'_Q[z, \ldots, z]$. By Proposition 2.9, $C'_Q[z, \ldots, z]$ contains at least one occurrence of z. If $C'_Q[z, \ldots, z] \not\equiv z$, then we have $M \equiv C[C_P[P, C_Q[Q]]] \to C[C_P[P, C'_Q[Q, \ldots, Q]]]$ where N has two essential subterms P and Q: contradiction. Thus, $C_Q[z]$ must collapse to z, that is, $C_Q[z] \to z$. Thus, we have $M \equiv C[C_P[P, C_Q[\underline{Q}]]] \twoheadrightarrow C[C_P[P\downarrow, \underline{Q}]]$. However, since $C[C_P[P\downarrow, z]]$ is normal form, we cannot obtain a reduction $C[C_P[P\downarrow, \underline{Q}]] \twoheadrightarrow \underline{Q}$ without erasing the underlining of $\underline{Q}$. This contradicts the fact that $\underline{Q}$ is an essential subterm of M. Hence, Case 2.1 is impossible.

Case 2.2. $M_Q \subset P$ (i.e., $Q \subset P$).

Let $M \equiv C[P]$. Then, from a similar reason as for Case 1, it follows that $C[z]$ must be a normal form. Since $P\downarrow$ has a white root, $M\downarrow \equiv C[P\downarrow]$. This contradicts the color change of M; hence, Case 2.2 is impossible.

Case 3. M_P and M_Q occur independently (i.e., $M_P \not\subseteq M_Q$, and $M_Q \not\subseteq M_P$).

Let $M \equiv C[M_P, M_Q]$, $M_P \equiv C_P[P]$, and $M_Q \equiv C_Q[Q]$. By a similar reason as for Case 1, $C[z, z']$ must be a normal form. If $C_P[z]$ and $C_Q[z]$ both are normal forms, then we have $M\downarrow \equiv C[C_P[P\downarrow], C_Q[Q\downarrow]]$: contradiction to the color change of M. Thus, both $C_P[z]$ and $C_Q[z]$ cannot be normal forms. Now consider the following two subcases:

Case 3.1. $C_P[z]$ is a normal form, but $C_Q[z]$ is not (or $C_Q[z]$ is a normal form but $C_P[z]$ is not).

Let $C_Q[z] \to C'_Q[z, \ldots, z]$. If $C'_Q[z, \ldots, z] \not\equiv z$, then $N \equiv C[C_P[P, C'_Q[Q, \ldots, Q]]]$ has two essential subterms P and Q by Proposition 2.9: contradiction. Thus, $C_Q[z] \to z$, and we have $M \equiv C[C_P[P, C_Q[\underline{Q}]]] \twoheadrightarrow C[C_P[P\downarrow, \underline{Q}]]$. However, since $C[C_P[P\downarrow, z]$ is a normal form, we cannot have $C[C_P[P\downarrow, \underline{Q}]] \twoheadrightarrow \underline{Q}$. This contradicts the fact that Q is an essential subterm of M. Hence, Case 3.1 is impossible.

Case 3.2. Neither $C_P[z]$ nor $C_Q[z]$ is a normal form.

By the same reason as for Case 3.1, we have $C_P[z] \to z$ and $C_Q[z] \to z$. Thus, we can write $M \equiv C[C_P[P], C_Q[Q]] \to N \equiv C[P, C_Q[Q]] \to N' \equiv C[P, Q]$. Note that since N has exactly one essential subterm, by Proposition 2.9 Q must be this unique essential subterm of N. Since N' has a black root and $N'\downarrow (\equiv M\downarrow)$ has a white root, N' must have at least one essential subterm, say P', by Lemma 3.7.2. However, we cannot take P nor Q as P' since P and Q are not special subterms of N'. Which special subterm of N' can we take as P'? If P' and Q occur independently in N', we can write $N \equiv C'[P', C_Q[Q]] \to N' \equiv C'[\underline{P'}, Q] \twoheadrightarrow \underline{P'}$. Hence, not only Q but also P' is

an essential subterm of N: contradiction to the uniqueness of essential subterm of N. If $P' \subset Q$, by a similar way we can easily show that P' is an essential subterm of N: again contradiction. If $Q \subset P'$, by Lemma 2.8 we have some essential subterm P'' in N such that $P'' \to P'$ and $Q \subset P''$: contradiction. Therefore, it follows that Case 3.2 is impossible. □

Appendix B

In this Appendix we prove termination of the direct sum of a left-linear complete TRS and a non-collapsing terminating TRS. A TRS is noncollapsing if it does not contain collapsing rewrite rules; a rewrite rule is a collapsing rule if its right-hand side is a variable.

Throughout this Appendix, let R_b and R_w be terminating TRSs. Rusinowitch [1987] proved:

THEOREM B.1. *If R_b and R_w are noncollapsing, then the direct sum $R_b \oplus R_w$ is terminating.*

In the present paper, we have shown:

THEOREM B.2. *If R_b and R_w are left-linear and confluent, then the direct sum $R_b \oplus R_w$ is terminating.*

These two facts suggest that the properties "*noncollapsing*" and "*left-linear and confluent*" are in some instances interchangeable. Thus, we are led to the following conjecture.

CONJECTURE B.3. *If R_b is noncollapsing and R_w is left-linear and confluent, then the direct sum $R_b \oplus R_w$ is terminating.*

We will show that this conjecture indeed holds. Henceforth, we assume that R_b is noncollapsing and terminating, and that R_w is left-linear and complete (i.e., confluent and terminating). Further, $\to_i$ $(i = b, w)$ denotes the reduction relation given by the rewrite rules of R_i. Note that a mixed term t has a unique normal form with respect to $\to_w$, denoted by $nf_w(t)$, because R_w is complete.

LEMMA B.4. *Let $t \to_b s$. Then we have $nf_w(t) \twoheadrightarrow_b nf_w(s)$. Moreover, if $t \to_b s$ is an outer reduction, then $nf_w(t) \to_b nf_w(s)$.*

PROOF. Since $t \to_b s$, there exists a rule $l \to r$ of R_b with the set of variables $\{x_1, \ldots, x_n\}$ occurring in l, a context $C[\ ]$, and mixed terms $t_1, \ldots, t_n$, such that $t \equiv C[l\sigma] \to_b s \equiv C[r\sigma]$, where the substitution σ is defined by

$$x_i\sigma \equiv t_i \ (i = 1, \ldots, n).$$

Let $nf_w(C[\ ]) \equiv C'[, \ldots,]$. By $x_i\sigma' \equiv nf_w(t_i)$ $(i = 1, \ldots, n)$ we define a new substitution σ'. From the noncollapsing property R_b of and the left-linearity of R_w, it follows that $nf_w(t) \equiv C'[l\sigma', \ldots, l\sigma']$ and $nf_w(s) \equiv C'[r\sigma', \ldots, r\sigma']$. Thus, $nf_w(t) \twoheadrightarrow_b nf_w(s)$ holds. If $t \to_b s$ is an outer reduction, then $nf_w(C[\ ]) \equiv C'[\ ]$, that is, C' contains precisely one hole. Hence, $nf_w(t) \to_b nf_w(s)$ follows. □

THEOREM B.5. *Let the TRSs R_b and R_w be terminating. If R_b is noncollapsing and R_w is left-linear and confluent, then the direct sum $R_b \oplus R_w$ is terminating.*

PROOF. By induction on rank(t) = k. The basic step $k = 1$ is trivial. Induction step: Assume the theorem for rank(t) < k. Then, we will show the case rank(t) = k. Suppose there exists an infinite reduction $t \to t' \to t'' \to \cdots$. From the induction hypothesis, all the terms t, t', t'', ... have the same rank. In this infinite reduction we must have infinitely many outer reduction steps; otherwise, we must have an infinite internal reduction, in contradiction with the induction hypothesis. Now consider the following cases.

Case 1. *root*(t) $\in \mathscr{F}_b$. *The infinite reduction* $t \to t' \to t'' \to \ldots$ *has the form*

$$\gamma: t \equiv t_0 \twoheadrightarrow_b t_1 \twoheadrightarrow_w t_2 \twoheadrightarrow_b t_3 \twoheadrightarrow_w t_4 \twoheadrightarrow_b \cdots.$$

Then by Lemma B.4 we have

$$\gamma': nf_w(t_0) \twoheadrightarrow_b nf_w(t_1) \equiv nf_w(t_2) \twoheadrightarrow_b nf_w(t_3) \equiv nf_w(t_4) \twoheadrightarrow_b \cdots.$$

Since the infinite reduction γ has infinitely many steps of outer reduction $\to_b$, by Lemma B.4 it follows that the reduction γ must have infinitely many $\to_b$ steps: contradiction.

Case 2. root(t) $\in \mathscr{F}_w$. By using outer reduction steps $\to_o$ and inner reduction steps $\to_i$, the infinite reduction can be presented as

$$\delta: t \equiv t_0 \twoheadrightarrow_i s_0 \twoheadrightarrow_o t_1 \twoheadrightarrow_i s_1 \twoheadrightarrow_o t_2 \twoheadrightarrow_i \cdots.$$

From the noncollapsing property of R_b, every inner reduction $t_n \twoheadrightarrow_i s_n$ in δ does not modify the outermost layer of t_n, that is, we can write $t_n \equiv C_n[\![t_1^n, \ldots, t_m^n]\!]$ and $s_n \equiv C_n[\![s_1^n, \ldots, s_m^n]\!]$ where $t_i^n \twoheadrightarrow s_i^n$ $(i = 1, \ldots, m)$. Hence, by replacing all the internal terms in δ with x, we obtain

$$\delta': C_0[x, \ldots, x] \twoheadrightarrow_w C_1[x, \ldots, x] \twoheadrightarrow_w C_2[x, \ldots, x] \twoheadrightarrow_w \cdots$$

where every $\twoheadrightarrow_w$ corresponds to $\twoheadrightarrow_o$ in δ. Since δ has infinitely many steps of $\to_w$ as $\to_o$, δ' must be an infinite reduction: contradiction. □

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